ON THE DYNAMICS OF THE PARAMETERS OF PULSES PROPAGATING IN THE MEDIUM WITH ANOMALOUS DISPERSION OF THE GROUP VELOCITY

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Abstract. Using the method of moments, a system of equations for the parameters of a pulse propagating in an isotropic medium with dispersion in the form of a Duhamel integral is obtained. A criterion has been found for the parameters of the pulse and the medium separating the propagation modes of soliton-like pulses.

Keywords: *dispersion, nonlinearity, light pulse*

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INTRODUCTION

When deriving model equations describing the propagation of laser pulses in nonlinear dispersing media, two well-known approximations are often used. The propagation of quasi-monochromatic pulses in a medium with Kerr nonlinearity in the dielectric transparency region can be described by the nonlinear Schrödinger equation (NSE) for the pulse envelope [1]. In this case, the corresponding criterion for a large number of field oscillations is given by the expression $(\omega_0\tau_p)^2 >> 1$, where ω_0 − is the center frequency of the pulse, τ_p is its duration. To describe pulses involving only a few oscillations of the light field $\omega_0\tau_p \sim 1$ (extremely short pulses), equations written directly for the electric field of the pulse or its spectrum are used [2-8]. For such pulses, the slowly varying envelope (SME) approximation becomes inapplicable. In both cases, when the pulse spectrum lies in the region of optical transparency, the corresponding condition can be written as $|\omega_0 - \omega_R|\tau_p >> 1$, where ω_R − is the characteristic frequency of the resonance absorption line. In this case, the expression for the dispersive response can be decomposed in a series. It is not difficult to see that for quasi-monochromatic pulses this relation coincides with the MMO condition at a significant distance of the carrier frequency of the pulse from the resonance. However, in the case of materials with several narrow absorption lines, for example, molecular gases, the condition of optical transparency can be violated even when the pulse contains a sufficiently large number of field oscillations, although no significant absorption occurs and the material can be considered transparent with a good degree of accuracy. The present work is devoted to the theoretical analysis of the equations describing the propagation of pulses in these cases and to finding the conditions of soliton-like modes.

THE METHOD OF MOMENTS

The equation describing unidirectional propagation of optical pulses in a nonlinear medium with dispersion has the following form

$$\frac{\partial E(z,\tau)}{\partial z} = -\frac{2\pi}{c}\frac{\partial}{\partial\tau}\left(\int_0^\infty \chi(\tau')E(z,\tau-\tau')d\tau' + \frac{\chi^{(3)}}{4\pi}E^3(z,\tau)\right) . (1)$$

where $\tau = t - z / c$, z is the coordinate along which the signal propagates, $\chi(\tau) = \Theta(\tau) 2e^2 \sum_{l,j} N_l A_{lj} \sin \omega_{lj}\tau / m\omega_{lj}$ is the impulse response function related to the dielectric susceptibility of the medium by Fourier transform $\chi(\omega) = \int_0^\infty \chi(\tau) e^{i\omega\tau} d\tau = (n_0 - 1) / 2\pi$,

$$n_0(\lambda) = 1 + \frac{4\pi e^2}{m\omega_0^2} \sum_{l,j} N_l A_{lj} \frac{\lambda_{lj}^2}{\lambda_0^2 - \lambda_{lj}^2}, \qquad (2)$$

$\Theta(\tau)$ – Heaviside function, e - electron charge, N_l - concentration of atoms or molecules of the variety , l A_{lj} - value proportional to the oscillator strength - of the j resonance, ω_{lj} - frequency of the corresponding resonance, c - speed of light in vacuum, λ_0 - central wavelength of the pulse, $\chi^{(3)} = \sum_l N_l \chi_l^{(3)} / \sum_l N_l$ -resulting cubic susceptibility of the medium, $n_2 = \sum_l N_l n_{2,l} / \sum_p N_l$ -resulting nonlinear refractive index, , $\chi_l^{(3)}$ $n_{2,l}$ -cubic susceptibility and nonlinear refractive index of atoms or molecules of the variety l . Let us represent the electric field E in the form

$$E(z,\tau) = \frac{1}{2}\psi(z,\tau)\exp(-i\omega_0\tau) + \kappa.c. \qquad (3)$$

Substituting (3) into (1), we obtain

$$\frac{\partial\psi}{\partial z} = -\frac{2\pi}{c}\int_0^\infty \frac{\partial\chi(\tau')}{\partial\tau'}\psi(z,\tau-\tau')e^{i\omega_0\tau'}d\tau' + i\gamma\psi|\psi|^2 - \frac{\gamma}{\omega_0}\frac{\partial}{\partial\tau}\left(\psi|\psi|^2\right). \qquad (4)$$

Here ψ is the envelope of the electric field, $\gamma = 3\chi^{(3)}\omega_0 / 8c = n_0^2\omega_0 n_2 / 8\pi$ is the cubic nonlinearity coefficient, $n_0 \approx 1 + 2\pi\chi$ is the refractive index of the medium, ω_0 is the center frequency of the signal. In the transition from equation (1) to (4), we neglected the generation of harmonics. In particular, in [9] it was shown that for pulses involving the order of one or two field fluctuations cubic nonlinearity causes the generation of the fourth harmonic. The corresponding effect of odd harmonic generation in a medium with quadratic nonlinearity was described in [10]. Note that equation (1) describes both quasi-monochromatic pulses and extremely short pulses [11-13].

The dynamics of the pulse parameters is analyzed on the basis of the method of moments [14]. Let us choose a trial solution in the form

$$\psi = B\exp\left(-\frac{1}{2}\left(\frac{\tau - T}{\tau_p}\right)^2 (1+iC) + i\left(\phi + \Omega\left(\tau - T\right)\right)\right), \tag{5}$$

where B is the signal amplitude, C is the parameter defining the frequency modulation, ϕ is the phase, Ω is the frequency shift. All parameters depend on the coordinate z . Let us define the moments of the pulse in the form

$$W = \int_{-\infty}^{\infty} |\psi|^2 \, d\tau \ , \tag{6}$$

$$\tau_p^2 = \frac{2}{W}\int_{-\infty}^{\infty} \left(\tau - T\right)^2 |\psi|^2 \, d\tau \ , \tag{7}$$

$$C = \frac{i}{W}\int_{-\infty}^{\infty} \left(\tau - T\right)\left(\psi^* \frac{\partial \psi}{\partial \tau} - \psi \frac{\partial \psi^*}{\partial \tau}\right) d\tau \, , \tag{8}$$

$$T = \frac{1}{W}\int_{-\infty}^{\infty} \tau |\psi|^2 \, d\tau \, , \tag{9}$$

$$\Omega = -\frac{i}{2W}\int_{-\infty}^{\infty} \left(\psi^* \frac{\partial \psi}{\partial \tau} - \psi \frac{\partial \psi^*}{\partial \tau}\right) d\tau \, , \tag{10}$$

phase can be found from the following expression

$$\int_{-\infty}^{\infty}\left(\frac{\partial\psi}{\partial z}\psi^{*}-\frac{\partial\psi^{*}}{\partial z}\psi\right)d\tau=-2i\sum_{l,j}D_{lj}\,\mathrm{Im}\int_{0}^{\infty}\int_{-\infty}^{\infty}\psi^{*}(\tau)\psi(\tau-\tau')e^{i(\omega_0-\omega_{lj})\tau'}d\tau d\tau'+$$

$$+2i\gamma\int_{-\infty}^{\infty}|\psi|^{4}d\tau-i\frac{\gamma}{\omega_0}\mathrm{Im}\int_{-\infty}^{\infty}|\psi|^{2}\psi^{*}\frac{\partial\psi}{\partial\tau}d\tau\,. \quad (11)$$

Using the method of moments, we obtain

$$W=W_0\exp\left(-2\,\mathrm{Im}\int_0^z k_s dz\right), \quad (12)$$

$$\frac{\partial\tau_p}{\partial z}=\mathrm{Im}\left[\frac{(1+iC)^2}{2\tau_p}\frac{\partial^2 k_s}{\partial\omega_0^2}\right], \quad (13)$$

$$\frac{\partial C}{\partial z}=\mathrm{Re}\left[\frac{(1+C^2)(1+iC)}{\tau_p^2}\frac{\partial^2 k_s}{\partial\omega_0^2}\right]+\frac{\gamma W}{\sqrt{2\pi}\tau_p}\left(1-\frac{\Omega}{\omega_0}\right), \quad (14)$$

$$\frac{\partial T}{\partial z}=\mathrm{Re}\left[(1+iC)\frac{\partial k_s}{\partial\omega_0}\right]+\frac{3\gamma W}{2\sqrt{2\pi}\omega_0\tau_p}, \quad (15)$$

$$\frac{\partial\Omega}{\partial z}=\mathrm{Im}\left[\frac{(1+C^2)}{\tau_p^2}\frac{\partial k_s}{\partial\omega_0}\right]-\frac{\gamma WC}{\sqrt{2\pi}\omega_0\tau_p^3}, \quad (16)$$

$$\frac{\partial\varphi}{\partial z}=\mathrm{Re}\left[k_s+\frac{(1+iC)^2}{4\tau_p^2}\frac{\partial^2 k_s}{\partial\omega_0^2}\right]+\frac{\gamma W}{4\sqrt{2\pi}\tau_p}\left(5+\frac{\Omega}{\omega_0}\right)+\Omega T_z\,. \quad (17)$$

Here $k_s=i\sum_{l,j}D_{lj}\tau_p\left(2iF(\zeta_{lj})+\sqrt{\pi}\exp(-\zeta_{lj}^2)\right)/\sqrt{1+C^2}$ is called the soliton

wave number, , , , ,$\zeta_{lj}=\tau_p\Delta\omega_{lj}/\sqrt{1+C^2}$ $\Delta\omega_{lj}=\omega_0-\Omega-\omega_{lj}$ $D_{lj}=2\pi e^2 N_l A_{lj}/mc$

$W = B^2 \tau_p \sqrt{\pi}$ $W_0 = B_0^2 \tau_0 \sqrt{\pi}$ B_0 τ_0 , ,are the initial values of the corresponding

parameters $F(\zeta) = \exp(-\zeta^2) \int_0^{\zeta} \exp(t^2) dt$ is the Dawson function.

SOLITON-LIKE PROPAGATION MODE

As a medium we will consider air, which consists of 21% oxygen O_2 and 79% nitrogen N_2 . Argon Ar , water vapor H_2O and carbon dioxide CO_2 account for less than one percent of the concentration of all molecules. The refractive index of air is presented in [15]. We consider the air transparency window belonging to the range from$3.5 - 4.1$ мкм , in which the group velocity dispersion β_2 is anomalous. The main contribution to the anomalous air dispersion is given by two resonant wavelengths of carbon dioxide ,$\lambda_1 = 4.223$ мкм$\lambda_2 = 4.291$ мкм , and therefore only these terms can be considered in the expression for the group dispersion coefficient$\beta_2 = (\lambda_0^3/2\pi c^2) d^2 n_0(\lambda_0)/d\lambda^2 = \partial^2 k/\partial\omega_0^2$ (where $k = \omega_0 n_0 / c$ is the wave number).

Oxygen and nitrogen give the largest contribution to the nonlinear refractive index of air [16]

$$n_2 = 0.79 n_{2,N_2} + 0.21 n_{2,O_2} \quad , n_{2,O_2,N_2} = \frac{P_{O_2,N_2}^{-1}}{\lambda_{O_2,N_2}^{-2} - \lambda_0^{-2}} \ . \ (18)$$

Here ,$P_{N_2} = 14.63$ ГВт$\lambda_{N_2} = 0.3334$ мкм for nitrogen and ,$P_{O_2} = 14.62$ ГВт$\lambda_{N_2} = 0.3360$ мкм for oxygen. The approximation (18) is valid in the range$1 - 4$ мкм .

To consider the soliton-like regime, we will put ,$C = 0 \partial C/\partial z = 0$. In addition, we will consider the limit

$$\Delta\omega\tau_p \geq 2.67 , \qquad (19)$$

where $,\Delta\omega = \omega_0 - \omega_1$ ω_1 is the resonant frequency of the medium nearest to the center frequency of the pulse. In this limit, the asymptotic series expansion of the Dawson function is valid [17]

$$F(x) = \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^5} + \frac{1\cdot 3\cdot 5\cdot ..(2n-1)}{2^{n+1}x^{2n+1}} \ . \qquad (20)$$

From the system (12) - (17) we obtain

$W = B_0^2 \tau_0 \sqrt{\pi} = 8\pi I_0 \tau_0 \sqrt{\pi} / cn_0, \Omega = 0$,

$$I_0 = \frac{c\sqrt{2}}{\tau_0^2 \omega_0 n_0 n_2} \left| \frac{\partial^2 k_{sR}}{\partial \omega_0^2} \right|, \qquad (21)$$

$$T = \left(\frac{\partial k_{sR}}{\partial \omega_0} + \frac{3 n_0 n_2 I_0}{2\sqrt{2}c} \right) z\, , \qquad (22)$$

$$\varphi = \left(k_{sR} + \frac{1}{4\tau_0} \frac{\partial^2 k_{sR}}{\partial \omega_0^2} + \frac{5 n_0 n_2 \omega_0 I_0}{4\sqrt{2}c} \right) z\, . \qquad (23)$$

$k_{sR} = \mathrm{Re}\, k_s$ - is the real part of the soliton wave number, the value$\partial^2 k_{sR}/\partial\omega_0^2$ can be called the soliton group dispersion coefficient. The dependence of the pulse intensity on its duration described by expression (21) is shown in Figs. *1a* and *1b*. For Fig. *1a*, condition (19) is satisfied if$\tau_p > 35$ фс , and for Fig. *1b* if .$\tau_p > 72$ фс

From equation (4), we can obtain an equation with variance as a series if we decompose the integrand function$\psi(z, \tau - \tau')$ into a series and use the Fourier transform $\partial^n \chi(\omega) / \partial \omega^n = (i)^n \int_0^\infty \tau^n \chi(\tau) e^{i\omega\tau} d\tau$.

Performing the transformations, we obtain

$$\frac{\partial \psi}{\partial z} = i\left(k - \frac{\omega_0}{c}\right)\psi - \sum_n \frac{(i)^{n-1}}{n!} \beta_n \frac{\partial^n \psi}{\partial \tau^n} + i\gamma\psi|\psi|^2 - \frac{\gamma}{\omega_0}\frac{\partial}{\partial \tau}\left(\psi|\psi|^2\right). \ (24)$$

Here, ,$\beta_1 = \partial k / \partial \omega_0 - 1/c$ $\beta_n = \partial^n k / \partial \omega_0^n$ ($n \ge 2$),$k = n_0 \omega_0 / c$. The solution of this equation coincides with (21) - (23) if the Dawson function is represented as an asymptotic series (20). Thus, the variance decomposition (24) is valid if condition (19) is satisfied. Otherwise, to describe the dynamics of pulses, we must consider equations (1) or (4).

Let us determine the applicability limits of the trial solution (5) using the rule of conservation of electric area [18]

$$\int_{-\infty}^{\infty} E(\tau, z) d\tau = const. \qquad (25)$$

Obviously, if the electric field of momentum can be represented as the time derivative $E = \partial \Phi / \partial \tau$ of a function decreasing at infinity, then condition (25) is satisfied and the momentum area is zero [4]. Let us represent the function Φ in the form [19]

$$\Phi(\tau, z) = -\frac{\psi}{2i\omega_0} \exp(-i\omega_0 \tau) + \kappa.c. \text{ , } (26)$$

then

$$E(\tau, z) = \frac{1}{2}\left(\psi \exp(-i\omega_0 \tau) - \frac{1}{i\omega_0} \frac{\partial \psi}{\partial \tau} \exp(-i\omega_0 \tau) + \kappa.c. \right) \qquad (27)$$

The contribution of the second summand in (27) is proportional to $1/\omega_0 \tau_p$ and can be neglected if the momentum includes about five or more field oscillations [19]. In this case, (27) goes to (3) with an envelope in the form of (5). Thus, the momentum area conservation rule imposes restrictions on the applicability of the trial solution of the form (5). It should be noted that the unidirectional propagation approximation should be used with caution because it can lead to violation of the electric area conservation rule [20-22].

CONCLUSION

The propagation of soliton-like pulses in air is analytically described using the method of moments. The dispersion contribution is taken into account by means of the Duhamel integral. The criterion (19) separating two modes of signal propagation is found. It is shown that the MMO approximation may be inapplicable for pulses including about ten field fluctuations if the pulse spectrum lies near the resonance of the medium. For these cases, a system of equations on the pulse parameters is obtained. A frequent solution of this system is found.

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FIGURE CAPTIONS

Fig. 1. Dependence of the pulse intensity on its duration at the central wavelength of the signal$\lambda = 3.6$ (a),3.9 мкм (b).

a *b*

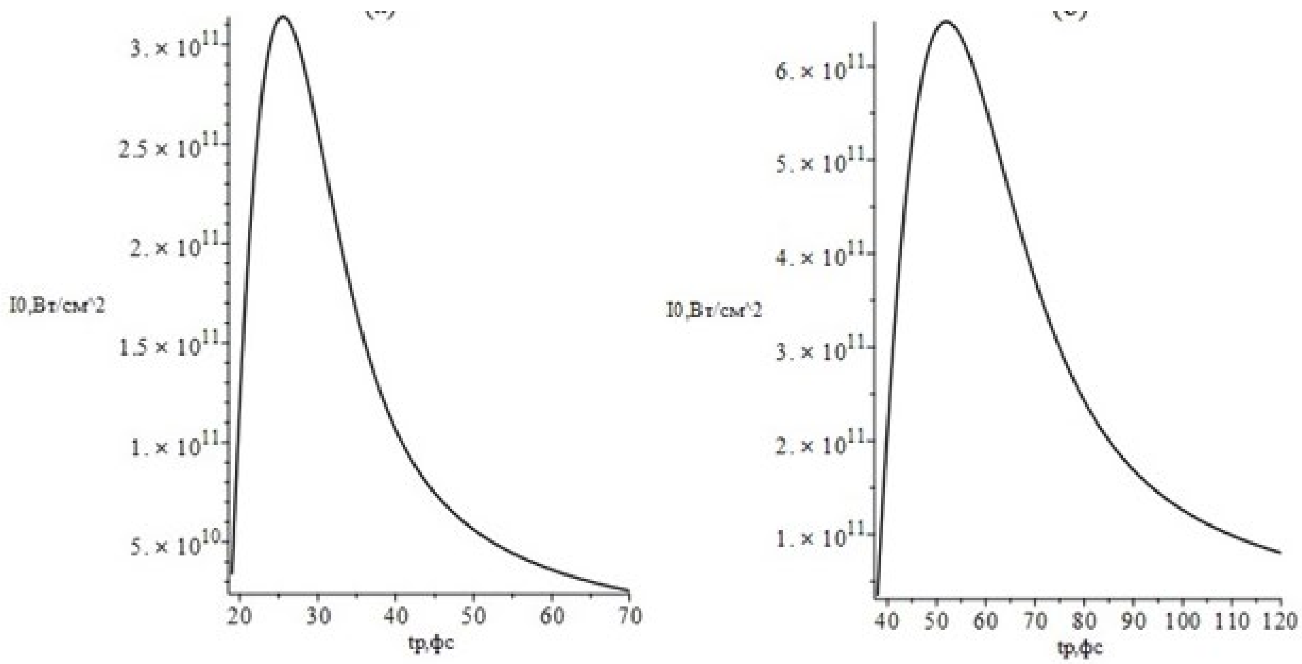

Fig. 1.