SOME CHARACTERISTICS OF NONLINEAR POTENTIAL SURFACE WAVES IN AN IDEAL LIQUID

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Abstract. The potential wave motion along the free surface of a deep ideal liquid is studied. Exact solutions of the equations of motion with physically justified boundary conditions on a free surface are obtained. The shape of the free surface is studied depending on the amplitude and the characteristics of the surface are highlighted. The characteristics of waves depending on the nonlinearity parameter are investigated

Keywords: *ideal fluid, potential flow, wave motion, Lambert W function, nonlinear waves, characteristics of wave motion*

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INTRODUCTION

Mankind has been attracted to the study of waves on the surface of liquids for quite a long time. Even Leonardo da Vinci in the fifteenth century mentioned surface waves in his works. With the advent of mathematical apparatus, research from simple observation to analytical description, and the development of experimental techniques has led to qualitatively new experimental descriptions of a variety of phenomena occurring on the surface of the liquid. Studies of surface waves find their place in classical treatises and textbooks [1-4] and in specialized monographs devoted to wave motions of liquids [5-8].

In modern works the linearized problem of calculating the characteristics of infinitesimal surface periodic flows in a liquid in various formulations is often solved [9 - 11]. Along with linearized problems, nonlinear waves become the object of study in theoretical works, and researchers in a wide variety of problem formulations obtain exact solutions of nonlinear Euler, Korteweg-de Fries, Schrödinger, and Hopf equations [12-16]. Interest in the study of surface waves has not died down until now, and they become the subject of both theoretical [12,13,16] and experimental [17-21] studies. The interest is related to the need to study the parameters of sea waves described in oceanography textbooks [22-23] for a detailed description and prediction of the behavior of wave and many related phenomena in the ocean. Despite the long history of the issue, a complete theory to predict wave behavior from measured characteristics has not yet been developed. In recent years, work on theoretical and experimental characterization has intensified. Experimental methods for controlling surface waves are also being developed (see, e.g., [24]).

To study and predict the behavior of waves at the sea surface, the registration of wave elements, characteristics and parameters is necessary. In the present work, a structurization of wave elements on the surface of the deep ocean described by Lambert functions is proposed. The solution of the problem in the form of Lambert functions was first obtained and discussed in [16]. The present work is aimed at investigating the influence of the wave amplitude on its various characteristics and experimentally measured parameters.

PROBLEM STATEMENT

Let us consider the propagation of periodic potential wave motions along the free surface of an ideal deep liquid occupying the lower half-space $z < 0$ in the Cartesian coordinate system. In the simplest case in the constant density approximation $\rho = const$ without taking into account surface tension in two-dimensional formulation, the mathematical formulation of the problem includes the Euler and continuity equations and is supplemented with standard boundary conditions on the free surface. When transferring to the coordinate system $O\xi z$, moving together with the wave with velocity c, taking into account the relationship between horizontal coordinates $\xi = x - ct$, the mathematical formulation of the problem is written as follows:

$$z < \zeta: \qquad \rho\left(-c\partial_\xi \vec{u} + \left(\vec{u}\nabla\right)\vec{u}\right) = -\nabla P + \rho\vec{g} \qquad (1)$$

$$\partial_t \rho + \vec{u}\cdot\nabla\rho + \rho\,\mathrm{div}\,\vec{u} = 0 \qquad (2)$$

$$z = \zeta: \qquad P = P_0, \qquad w - u\partial_\xi\zeta = -c\partial_\xi\zeta \qquad (3)$$

Here, the function $\zeta = \zeta(\xi)$ describes the deviation of the free surface from the equilibrium position $z = 0$, the free-fall acceleration $\vec{g} = (0, -g)$ is directed vertically downward against the direction of the axis Oz, the variables u and w denote the horizontal and vertical velocity components $\vec{u}$, and the pressure P is composed of atmospheric P_0, hydrostatic and dynamic pressure : $\tilde{P}$

$$P(\xi,z)=P_0+\rho g(\zeta-z)+\tilde{P}(\xi,z) \tag{4}$$

For a fluid with constant density, the continuity equation is reduced to the incompressibility condition, and in the two-dimensional formulation we can introduce a current function ψ such that $\vec{u}=(u,w)=(\partial_z\psi,-\partial_x\psi)$. The solution for the current function is sought in the form of a representation defining the exponential decay of wave motion with distance from the free surface and the representation generalized for the wave packet is as follows:

$$\psi(\xi,z)=\int_0^\infty \exp\big(k\big(z-\zeta(\xi)\big)\big)\phi(k,\xi)\,dk \tag{5}$$

From the kinematic boundary condition (3) and (5) we obtain for the current function:

$$\begin{gathered}\psi(\xi,z)=c(\zeta+a)\int_0^\infty \exp\big(k\big(z-\zeta(\xi)\big)\big)f(k)\,dk\\ \int_0^\infty f(k)\,dk=1\end{gathered} \tag{6}$$

Without going into the specifics of the solution, which is not the main interest of the present consideration and is considered and described in detail in [13], we note that for the function describing the deviation of the free surface from the equilibrium value we obtain the expression:

$$\zeta(\xi,A)=-\frac{1}{k}\left(W\left(-\frac{kA}{2}\exp(ik\xi)\right)+W\left(-\frac{kA}{2}\exp(-ik\xi)\right)\right) \tag{7}$$

Here, the symbol A denotes the amplitude of the wave motion, and $W(x)$ -W is the Lambert function.

In [16] it is shown that the function $f(k)$ in expression (6) is the Dirac delta function $\delta(k-k_*)$ and in that case, applying the Lambert function property $W(x)\exp(W(x))=x$ can be written for the current function:

$$\psi_\pm(\xi,z)=-\frac{c}{k_*}W\left(-k_*A\,\mathrm{e}^{\pm ik_*\xi}\right)\exp\left(W\left(-k_*A\,\mathrm{e}^{\pm ik_*\xi}\right)\right)\mathrm{e}^{k_*z}=cA\,\mathrm{e}^{\pm ik_*\xi}\,\mathrm{e}^{k_*z} \tag{8}$$

The present work is devoted to the analysis of the shape and some characteristics of the free surface, in the problem of surface wave propagation, the exact solution of which is defined by Lambert waves.

FREE SURFACE SHAPE ANALYSIS

Let us analyze the behavior of the free surface shape (7). For infinitesimal waves $kA \ll 1$ and the expression describing the shape of the free surface (), taking into account the decomposition of the Lambert function by the small parameter $W(x)=x-x^2+\frac{3}{2}x^3+o(x^3)$ at $|x| \ll 1$, takes the form:

$$\begin{aligned}\zeta(\xi,A)&=-\frac{1}{k}\left(W\left(-\frac{kA}{2}\exp(ik\xi)\right)+W\left(-\frac{kA}{2}\exp(-ik\xi)\right)\right)=\\&=-\frac{1}{k}\left(-\frac{kA}{2}\exp(ik\xi)-\frac{kA}{2}\exp(-ik\xi)\right)=A\cos(k\xi)\end{aligned} \tag{9}$$

Thus, at small amplitudes the shape of the free surface is close to harmonic. Let us plot the shape of the free surface for different nonlinearity parameters $\varepsilon = A\omega^2/g$, characterizing the ratio of the wave motion amplitude A to the wavelength. With the growth of the nonlinearity parameter the crest of the wave is stretched out and at some critical value $\varepsilon_{cr}=2/\mathrm{e}$ a singularity appears at the top of the wave and the surface shape ceases to be smooth. The waves satisfying the condition $\varepsilon < \varepsilon_{cr}$ will be called smooth or pre-critical waves, and the waves satisfying $\varepsilon \geq \varepsilon_{cr}$ will be called sharp or subcritical. Figure 1a shows typical surface shapes for precritical and zacritical waves. Surface waves are characterized by a large set of parameters that can be monitored in experimental studies. Figs. 1b and 1c show the surface shapes for smooth and sharp waves, indicating the characteristics of surface wave motion, the estimation of which can be obtained by processing optical wave data on the surface of fluid. Table 1 lists the characteristics and describes them. It is interesting that some parameters in the transition from the subcritical to the subcritical region lose their meaning, and some, on the contrary, appear more clearly. As an example, the angle between the lines of the steepest shear and wave front γ and the level of the midline ζ_s , which determines the position above and below which the areas swept by the wave are the same. The angle γ for subcritical waves is the same as the angle β at the top of the wave crest, while these angles are different for subcritical waves. The position of the midline ζ_s for precritical waves is close to the equilibrium position $z = 0$, however, for subcritical waves this level is noticeably higher than the equilibrium level.

Let us consider how the increase of the wave amplitude affects some characteristics of surface waves. In Fig. 2 we plot the dependences of the duration of the wave section, related to its length, exceeding the equilibrium level (having positive values of abscissas) $\lambda_{\lambda+} = \lambda_+/\lambda$, not exceeding (having negative values of abscissas) $\lambda_{\lambda-} = \lambda_-/\lambda$ and their difference $\Delta\lambda_\lambda = \lambda_{\lambda-} - \lambda_{\lambda+} = \left(\lambda_- - \lambda_+\right)/\lambda$. For convenience, the plots are made in values related to the wavelength for different nonlinearity parameters ε . At small amplitudes, the wavelengths above and below the equilibrium level are almost identical, however, as the amplitude increases, the section above the equilibrium level shortens compared to the section below the equilibrium level. Although the duration $\lambda_{\lambda+}$ decreases with increasing amplitude, the free surface length of the wave behaves in a nonlinear manner. The length of the free surface together with the position of the level ζ_s characterizes the potential energy: an increase in these quantities leads to an increase in the potential energy of the wave motion.

Fig. 3 shows the dependences on the nonlinearity parameter of the surface lengths of the wave sections related to the wavelength for the section exceeding the equilibrium level L_+/λ (Fig. 3a), not exceeding the equilibrium level L_-/λ (Fig. 3b) and the total free surface length $L/\lambda = (L_+ + L_-)/\lambda$. It can be seen that for pre-critical smooth waves with increasing amplitude there is a smooth insignificant increase in the surface length of the wave section below the equilibrium level and a relatively sharp increase in the surface length of the section above the equilibrium level. As the critical value of the nonlinearity parameter is exceeded, there is a less abrupt decrease in the free surface length of this section. Near the critical value of the nonlinearity parameter, there is a region of parameters for which the free surface length of the upper part of the wave exceeds the free surface length in the lower part of the wave. The same region is characterized by the largest surface length of the wave (see Fig. 3c) and, consequently, by the largest available surface potential energy.

Consider the angles characterizing the free surface of the wave. Fig. 4 shows the dependences of the angles characterizing different parts of the wave depending on the nonlinearity parameter. Fig. 4a presents the dependence of the angle at the bottom of the wave α . With increasing wave amplitude, the angle at the sole of the wave practically does not change and is close to $180°$. For the angle at the top of the wave

β (see Fig. 4b), a similar trend is observed only for smooth pre-critical waves; however, as the nonlinearity parameter approaches the critical value, there is a sharp decrease in the value of the angle and a subsequent smooth growth with increasing amplitude for subcritical waves. The angle between the lines with the maximum steepness of the front and the cutoff γ for zacritic waves coincides with the angle at the top β , and in the region of pre-critical values of the nonlinearity parameter there is a smoother decrease in the value compared to the angle β . The angle formed by the lines connecting the top of the wave with two neighboring soles φ (or between the lines connecting the soles with two neighboring vertices) also decreases with increasing amplitude in the region of pre-critical values of the nonlinearity parameter, reaches a minimum value (about $100°$) at the critical value, and for subcritical waves smoothly increases with increasing amplitude.

The described characteristics, as well as other quantities depicted in Fig. 1b, c can be tracked when setting up experiments to determine the parameters of the wave motion, the exact solution of which is described by Lambert functions.

CONCLUSION

The solution of the nonlinear problem of gravitational wave propagation along the free surface of an ideal deep incompressible fluid is obtained. Wave parameters, which can be used in the experiment, to characterize the waves, are constructed. With the increase of the amplitude of the wave motion there is a sharpening of the wave tops and at some critical value of the amplitude a special point appears at the top. The critical value of amplitude differentiates smooth pre-critical waves from sharpened subcritical waves. The effect of varying the amplitude of wave motion on the duration and surface length of Lambert wave sections for the precritical and zacritical amplitudes is investigated. The angles formed by tangents to the free surface at the top, bottom of the wave, as well as at points on the shear and front of the wave characterized by maximum values of steepness are investigated. The coordinates of these positions are obtained and it is shown that for subcritical waves these coordinates coincide with the coordinates of the wave top. The proposed description is given in observed and possible values for fixing in the experiment.

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FIGURE CAPTIONS

Fig. 1. Surface wave profiles: general view for different nonlinearity parameters of pre-critical, critical, and subcritical (a); wave with pre-critical nonlinearity parameter with indication of wave motion characteristics (b); wave with subcritical nonlinearity parameter with indication of wave motion characteristics (c).

Fig. 2. Wavelength-weighted duration of the wave segment as a function of the nonlinearity parameter above the equilibrium level $\lambda_{\lambda+}$, below the equilibrium level $\lambda_{\lambda+}$, and the difference in duration below the equilibrium level and above the equilibrium level . $\Delta\lambda_{\lambda} = \lambda_{\lambda-} - \lambda_{\lambda+}$

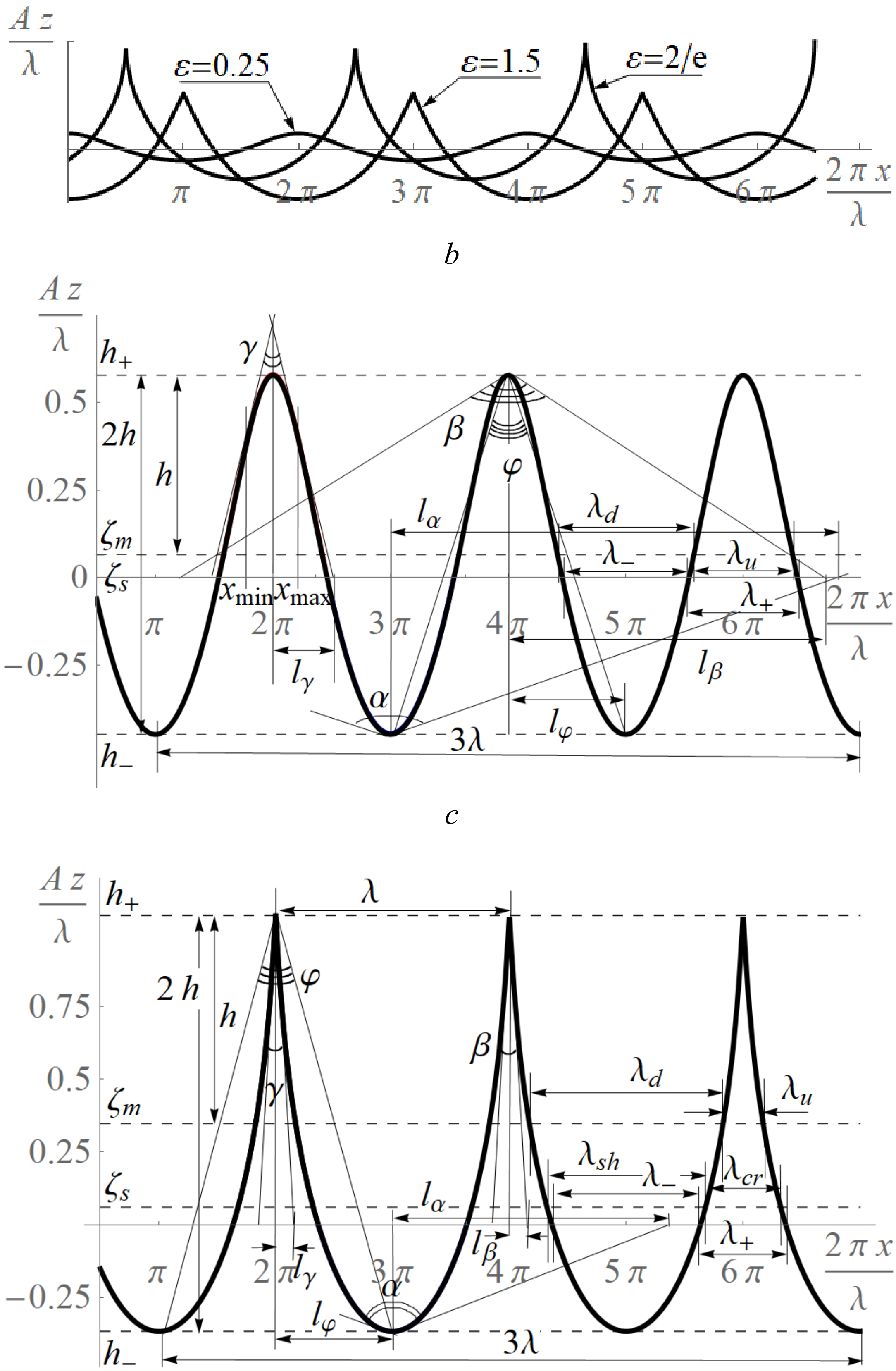
Fig. 3. Relative surface length of the wave section as a function of the nonlinearity parameter above the equilibrium level L_{+}/λ (a); below the equilibrium level L_{-}/λ (b). Total length of the surface section L/λ (c); relative elongation of the free surface $(L-\lambda)/\lambda$ (d).

Fig. 4. Characteristic angles of the wave as a function of the nonlinearity parameter: angle at the sole α (a), angle at the top β (b), angle between the lines with maximum steepness of the front and cutoff γ (c), angle between the lines connecting the top and neighboring soles φ (d).

Table 1. Characteristics of the surface wave disturbance.

Designation	Characterization
λ	Wavelength
A	Wave amplitude
h_+, h_-	Elevation and depth of the wave
ζ_s	Mid-wave level, the level that cuts off equal areas and characterizes the position of the center of mass
ζ_m	The middle level defining the position equidistant from the summit and the bottom
$\lambda_{sh}, \lambda_{cr}$	Duration of the trough and crest of the wave
λ_d, λ_u	Duration of the section below and above the average wave level ζ_m
λ_-, λ_+	Duration of the wave segment below and above the equilibrium position $z = 0$
α	Angle at the bottom of the wave
β	Angle at the top of the wave
γ	Angle between lines with maximum steepness of front and cutoff
φ	The angle between the lines connecting the top and adjacent soles, characterizing the steepness of the wave
$x_{\max}$, $x_{\min}$	Values of abscissas, at which the maximum steepness of the wave front and cutoff is reached

a



A z / λ
ε=0.25
ε=1.5
ε=2/e
π
2 π
3 π
4 π
5 π
6 π
2 π x / λ
b
A z / λ
h+
γ
2h
h
β
φ
0.5
0.25
ζm
0
ζs
xmin
xmax
lα
λd
λ−
λu
λ+
π
2 π
3 π
4 π
5 π
6 π
−0.25
lγ
α
lφ
lβ
3λ
h−
2 π x / λ
c
A z / λ
h+
λ
2 h
h
φ
0.75
β
0.5
λd
λu
ζm
γ
0.25
λsh
λ−
λcr
ζs
lα
lβ
λ+
π
2 π
lγ
3 π
α
4 π
5 π
6 π
2 π x / λ
−0.25
lφ
3λ
h−

Fig. 1.

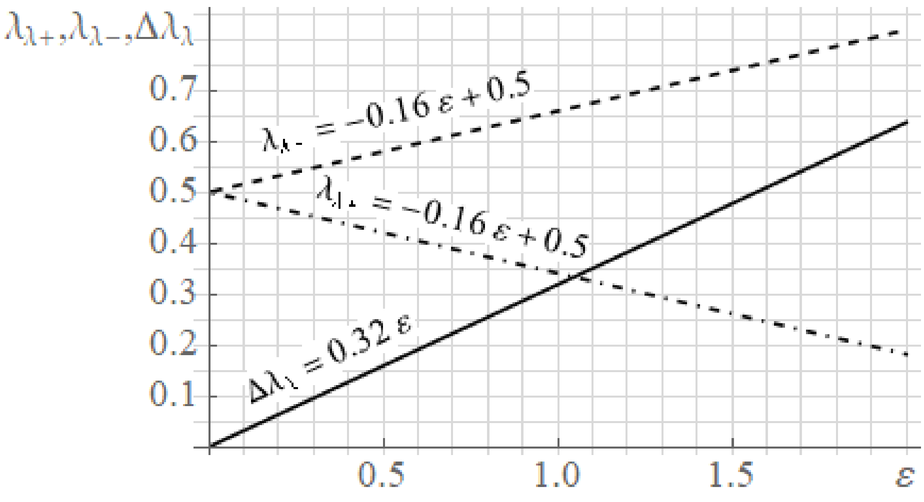
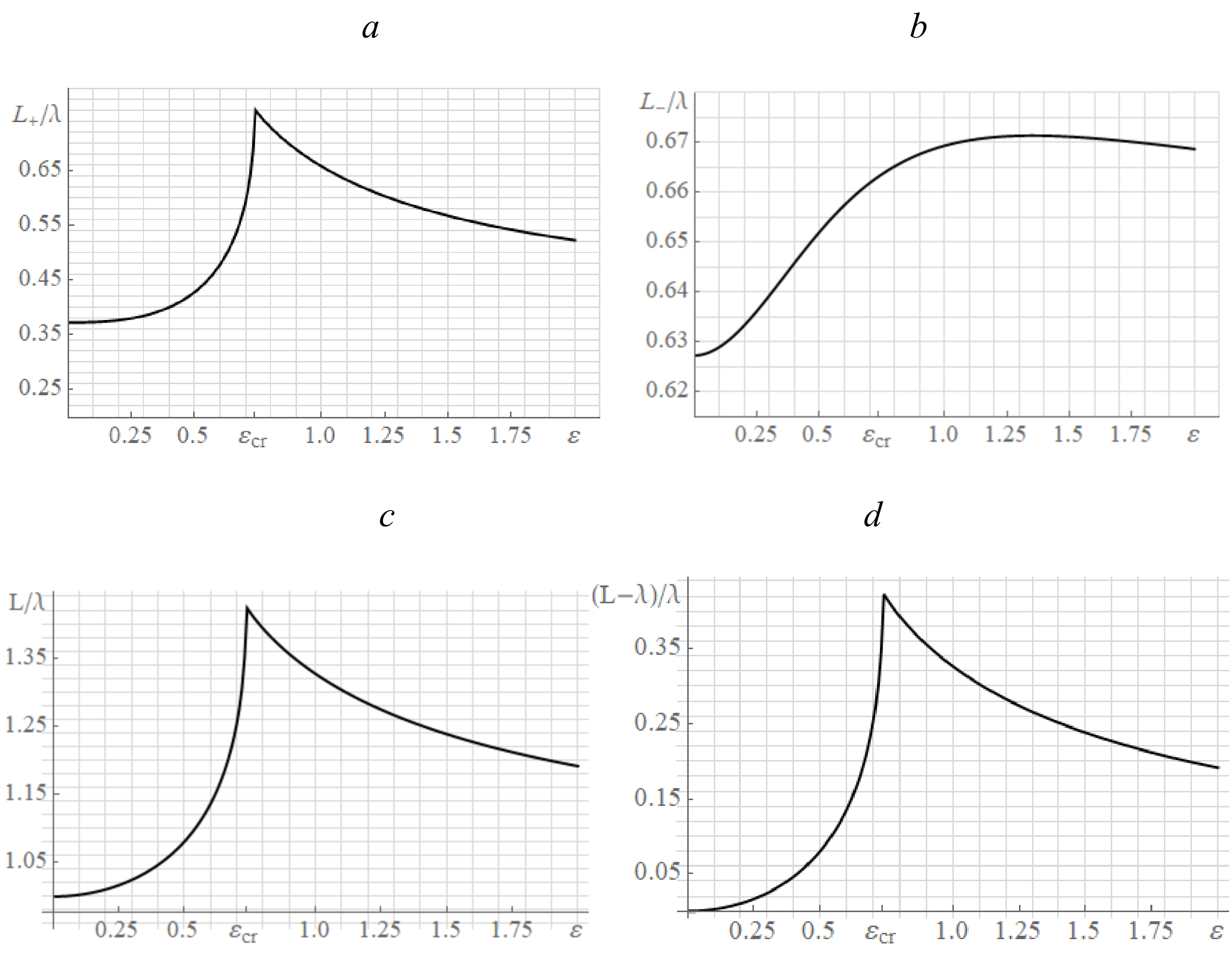

λλ+,λλ−,Δλλ
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.5
1.0
1.5
ε
λλ- = −0.16 ε + 0.5
λλ+ = −0.16 ε + 0.5
Δλλ = 0.32 ε

Fig. 2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b
L+/λ
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0.55
0.45
0.35
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0.25
0.5
εcr
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1.25
1.5
1.75
ε
L−/λ
0.67
0.66
0.65
0.64
0.63
0.62
c
d
L/λ
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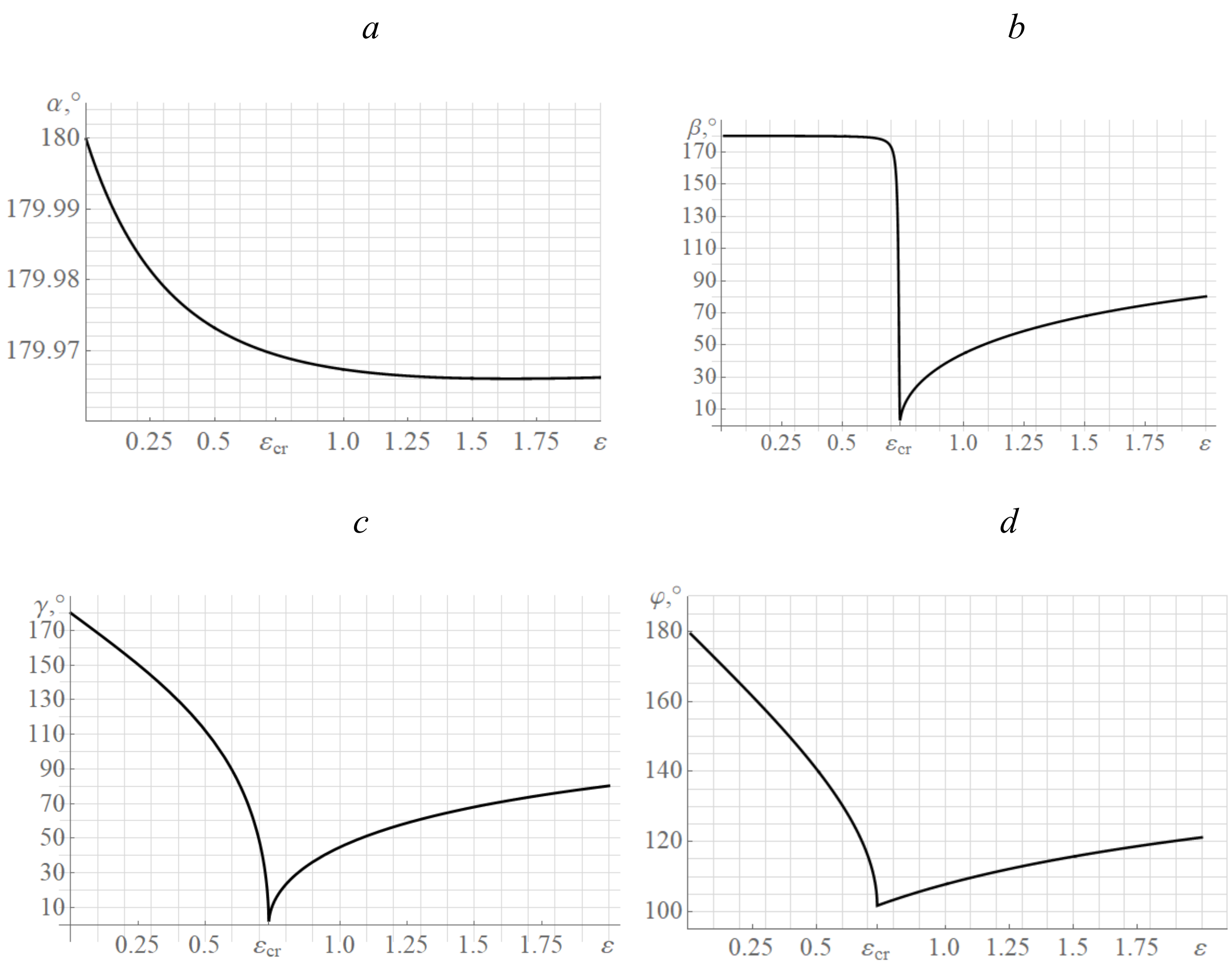
1.25
1.15
1.05
(L−λ)/λ
0.35
0.25
0.15
0.05

Fig. 3.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b
c
d
α,°
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179.98
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0.25
0.5
ε_cr
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1.25
1.5
1.75
ε
β,°
170
150
130
110
90
70
50
30
10
γ,°
φ,°
160
140
120
100

Fig. 4.