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ON WELL-POSEDNESS OF GENERALIZED NEURAL FIELD EQUATIONS WITH IMPULSIVE CONTROL

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We formulate and prove the theorem on well-posedness of abstract Volterra equations in metric spaces. We consider nonlinear nonlocal integral Volterra equation involving generalizing equations typically used in mathematical neuroscience. We investigate solutions that tend to zero at any moment with unbounded growth of the spatial variable. In the literature such solutions are called «spatially localized solutions» or «bumps». They correspond to normal brain functioning. For the main equation, we consider an impulsive control problem, where the control parameters are moments, when the solution discontinue, and corresponding jumps' values. Such control systems model electrical stimulation used in the presence of various diseases of central nervous system. We define suitable complete metric (not linear) space. Using the aforementioned theorem, we obtain conditions for existence and uniqueness of solution to this equation and continuous dependence of this solution on the control.
Key words: abstract Volterra equations; nonlinear integral equations; neural field equations; impulsive control; well-posedness.

1. Introduction

Unique solvability and continuous dependence of the solution on the model parameters has been always considered as important properties of a model, conditioning, in particular, its applicability, possibility to implement various numerical methods. The problem of continuous dependence on parameters of various classes of operator equations has been considered in numerous papers (see, e.g. the review [1] as well as the monograph [2] (pp. 203–210) and the references therein).

The present work extends the results of [21] on solvability of abstract Volterra equations in metric spaces by formulating and proving a statement on continuous dependence of solutions to these equations on a parameter. Utilizing these general results, we investigate well-posedness (existence, uniqueness and continuous dependence of solution on parameters) of an integral Volterra equation describing a broad class of models arising in neural field theory. Typical representative of this class is the Amari neural field equation [3]

$$w_t(t,x) = -w(t,x) + \int\limits_R \Omega(x-y) f(w(t,y))dy, \ t \geq 0, x \in \mathbb{R} \qquad (1.1)$$

Here the function w is the unknown variable, whose value $w(t,x)$ denotes the activity of the neural element at time t and position x, the non-negative function f gives the firing rate $f(w)$ of a neuron with activity w. The connectivity function (spatial convolution kernel) Ω determines the coupling strength.

The literature on the model (1.1) and its extensions is rather vast (see, e.g. [4]–[7]). The key issues in most of the published papers on these models are existence and stability of so-called bump-solutions, i.e. solutions satisfying the following condition

$$\lim_{|x|\to\infty} w(t,x)=0,\ t\in[a,\infty). \tag{1.2}$$

This type of solutions corresponds to the electrical brain activity that is prevalent during its normal functioning, encoding visual stimuli [8], representing head direction [9], and maintaining persistent activity states in working memory [10], [11].

The models of the type (1.1) are important in studies of cortical gain control or pharmacological manipulations [12]. The problems of therapy of Epilepsy, Parkinson's disease, and other disorders of the central nervous system has been recently investigated in [13]–[17]. The modeling frameworks in [13]–[17] incorporate brain electrical stimulation, which is considered as control, and the corresponding optimization problems. Here we model this electrical stimulation by means of impulsive control imposed on the variable w and investigate well-posedness of such models. Namely, we generalize the models considered in [3] – [5], [7] by introducing the following neural field equation

$$w(t,x)=\int_a^t\int_{\mathbb{R}^m} f(t,s,x,y,w(s,y))dyds+\sum_{k=0}^{\infty}\chi_{[t_k,\infty)}(t)u_k(x),\ t\in[a,\infty),\ x\in\mathbb{R}^m, \tag{1.3}$$

with respect to the unknown function $w\colon[a,\infty)\times\mathbb{R}^m\to\mathbb{R}^n$, which satisfies (1.2). Here, the function $f\colon\triangle\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$, $\triangle=\{(t,s)\in[a,\infty)\times[a,\infty),\ s\le t\}$ is given; the points t_k, $a\le t_1<$ $<t_2<\ldots$, and the functions $u_k\colon\mathbb{R}^m\to\mathbb{R}^n$, $k=1,2,\ldots$, are control parameters (for the sake of notational convenience, we put $t_0=a$, $u_0(x)\equiv 0$ for all $x\in\mathbb{R}^m$); χ_M denotes the characteristic function of the set $M\subset[a,\infty)$.

We obtain results on unique solvability of (1.3), (1.2) and continuous dependence of the solution on control $\mathfrak{U}=\{(t_k,u_k),\ k=1,2,\ldots\}$.

If we consider the unknown variable in (1.3) as a mapping $t\in[a,\infty)\mapsto w(t,\cdot)$, than the equation (3) can be formalized in terms of Volterra operator equation in the corresponding functional space. Volterra equations are usually considered in spaces possessing linear structure (see e.g. [18] – [20] and the references therein). However, if the impulse moments t_k are not fixed, and their amount is uniformly bounded by some given number on each given time interval, the set of functions of the kind $\sum\limits_{k=0}^{\infty}\chi_{[t_k,\infty)}(t)u_k(x)$ is not closed with respect to addition. This fact makes impossible the application of the theory developed in [18] – [20] to the problem (1.3), (1.2).

We would like to emphasize that the investigation of (1.3), (1.2) is not possible in the frameworks of [20], [19] due to their suitability only for the spaces possessing linear structure.

2. Existence, uniqueness and continuous dependence on parameters of solutions to abstract Volterra operator equations in metric spaces

We cite here important notions and results of the work [21] on solvability of abstract Volterra operator equations and extend the results of the paper [18] on continuous dependence on parameters of solutions to abstract Volterra equations to metric spaces.

Let us introduce the following notation:

$\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$;

$\mathbb{R}^m$ is the m-dimensional real vector space with the norm $|\cdot|$;

W and Λ are some metric spaces with the distances ρ_W and ρ_Λ, respectively;

$B_W(w,r)$ is a closed ball of the radius r centered at $w\in W$, $B_W^{out}(w,r)=W\setminus B_W(w,r)$;

μ is the Lebesgue measure on $\mathbb{R}^m$;

$L([a,b],\mu,\mathbb{R}^n)$ is the space of all measurable and integrable functions $\zeta:[a,b]\to\mathbb{R}^n$ with the norm $\|\zeta\|_{L([a,b],\mu,\mathbb{R}^n)}=\int\limits_\Omega|\zeta(s)|ds$;

$C_0(\mathbb{R}^m,\mathbb{R}^n)$ is the space of all continuous functions $\vartheta:\mathbb{R}^m\to\mathbb{R}^n$ satisfying the additional condition $\lim\limits_{|x|\to\infty}\vartheta(x)=0$, with the norm $\|\vartheta\|_{C_0(\mathbb{R}^m,\mathbb{R}^n)}=\max\limits_{x\in\mathbb{R}^m}|\vartheta(x)|$.

Let an equivalence relation $\sim$ be defined on W. For any two equivalence classes $\overline{w}^1$, $\overline{w}^2$, we introduce

$$d(\overline{w}^1,\overline{w}^2)=\inf_{w^1\in\overline{w}^1,\ w^2\in\overline{w}^2}\rho(w^1,w^2). \tag{2.1}$$

If for any $\varepsilon>0$ and any $\overline{w}^1,\overline{w}^2\in W/\sim$, $w^1\in\overline{w}^1$ one can find $w^2\in\overline{w}^2$ such that $d(\overline{w}^1,\overline{w}^2)=$ $=\rho(w^1,w^2)-\varepsilon$, then (2.1) defines metric in $W/\sim$.

We put in correspondence to each $\gamma\in[0,1]$ the equivalence relation $\upsilon(\gamma)$. We assume that the family of equivalence relations $\upsilon=\{\upsilon(\gamma),\gamma\in[0,1]\}$ satisfy the following conditions:

υ_0) $\gamma=0$ corresponds to the relation $\upsilon(0)=W^2$ (any two elements are $\upsilon(0)$-equivalent);

υ_0) $\gamma=1$ corresponds to equality relation (any two distinct elements are not $\upsilon(0)$-equivalent);

υ) if $\gamma_1>\gamma_2$, then $\upsilon(\gamma_1)\subseteq\upsilon(\gamma_2)$ (any $\upsilon(\gamma_1)$-equivalent elements are $\upsilon(\gamma_2)$-equivalent);

D e f i n i t i o n 1. [21] An operator $\Psi:W\to W$ is said to be a *Volterra operator on the family* υ if for any $\gamma\in[0,1]$ and any $w^1,w^2\in W$ the fact that $(w^1,w^2)\in\upsilon(\gamma)$ implies $(\Psi w^1,\Psi w^2)\in\upsilon(\gamma)$.

For any $w\in W$, let us denote $\overline{w}_\gamma$ to be an equivalence class of w.

Hereinafter it is assumed that (W,ρ_W) is a complete metric space with the equivalence relation υ satisfying $\upsilon_0),\upsilon_1),\upsilon)$. Moreover, we assume that for each $\gamma\in(0,1)$, the corresponding equivalence class is closed and the quotient set $W/\upsilon(\gamma)$ is a complete metric quotient space with the distance $d_{W/\upsilon(\gamma)}(\overline{w}^1,\overline{w}^2)=\inf\limits_{w^1\in\overline{w}^1,\ w^2\in\overline{w}^2}\rho_W(w^1,w^2)$.

We also cite some important properties of Volterra operators implied by Definition 1.

1. Choose an arbitrary set $\Gamma\subset[0,1]$, $\{0,1\}\subset\Gamma$, and for any decreasing (or any increasing) sequence $\{\gamma_i\}$, it holds true that $\lim\limits_{i\to\infty}\gamma_i\in\Gamma$. Let $\omega=\{\upsilon(\gamma),\gamma\in\Gamma\}$. We define the mapping $\eta:[0,1]\to\Gamma$ as $\eta(\gamma)=\inf\{\xi\in\Gamma\},\xi\geq\gamma$ $(\eta(\gamma)=\inf\{\xi\in\Gamma\},\xi\leq\gamma)$, and put in correspondence to any γ the equivalence relation $\upsilon(\eta(\gamma))$. If the operator $\Psi:W\to W$ is a Volterra operator on the family υ, then it is a Volterra operator on its subfamily ω.

2. Any composition of Volterra operators on a fixed family possesses the property 1.

3. The identity operator is a Volterra operator on any family of equivalence relations.

4. If for some $\gamma_0\in(0,1)$, $w\in W$ it holds true that $\Psi w\in\overline{w}_{\gamma_0}$, then the set $\overline{w}_{\gamma_0}$ is invariant with respect to the Volterra operator $\Psi:W\to W$ and the relation $\upsilon(\gamma)$ can be considered only on the elements of $\overline{w}_{\gamma_0}\subset W$. The set $\overline{w}_{\gamma_0}$ is a complete metric space with respect to the metric of the whole space W. Thus, the family of the equivalence relations satisfying the conditions $\upsilon_0,\upsilon_1,\upsilon$ is also defined on $\overline{w}_{\gamma_0}$. The quotient set $\overline{w}_{\gamma_0}/\upsilon(\gamma)$, $\gamma\leq\gamma_0$, consists of the unique element. If $\gamma>\gamma_0$, the quotient set $\overline{w}_{\gamma_0}/\upsilon(\gamma)$ is a complete metric space. Moreover, the fact that $\Psi:W\to W$ is a

Volterra operator on the family υ implies that the restriction $\Psi_{\gamma_0}:\overline{w}_{\gamma_0}\to\overline{w}_{\gamma_0}$ of Ψ is a Volterra operator on the family υ.

5. For each $\gamma\in(0,1)$, we define the canonical projection $\Pi_\gamma:W\to W/\upsilon(\gamma)$ as $\Pi_\gamma w=\overline{w}_\gamma$. For a Volterra operator $\Psi:W\to W$ on the family υ, we define the operator $\Psi_\gamma:W/\upsilon(\gamma)\to W/\upsilon(\gamma)$ as $\Psi_\gamma\overline{w}_\gamma=\Pi_\gamma\Psi w$, where w is an arbitrary element of $\overline{w}_\gamma$. Choose an arbitrary $\gamma_0\in(0,1)$. The family $\upsilon(\gamma_0)$ generate the corresponding equivalence relation on $W/\upsilon(\gamma_0)$. Let $\xi\in(0,\gamma_0)$, and let the elements $w^1,w^2\in W$ be $\upsilon(\xi)$-equivalent. Then any $w^{1'}\in\overline{w}^1_{\gamma_0}$, $w^{2'}\in\overline{w}^2_{\gamma_0}$ are also $\upsilon(\xi)$-equivalent, which defines the notion of equivalence of the classes $\overline{w}^1_{\gamma_0}$ and $\overline{w}^2_{\gamma_0}$. Namely, the classes $\overline{w}^1_{\gamma_0}$ and $\overline{w}^2_{\gamma_0}$ are $\overline{\upsilon}_{\gamma_0}(\sigma)$-equivalent ($\sigma\in(0,1)$), if there exist (which, actually, means «any») $w^1\in\overline{w}^1_{\gamma_0}$, $w^2\in\overline{w}^2_{\gamma_0}$ satisfying the equivalence relation $\upsilon(\xi)$, $\xi=\gamma_0\sigma$. Thus, the family $\overline{\upsilon}_{\gamma_0}=\{\overline{\upsilon}_{\gamma_0}(\sigma)\}$ of equivalence relations is defined on $W/\upsilon(\gamma_0)$. The quotient set $\Big(W/\upsilon(\gamma_0)\Big)/\overline{\upsilon}_{\gamma_0}$ with the distance

$$d(W_{\gamma_0\sigma},W_{\gamma_0\sigma})=\inf_{\overline{w}^1_{\gamma_0}\in W_{\gamma_0\sigma},\overline{w}^2_{\gamma_0}\in W_{\gamma_0\sigma}} d_{W/\upsilon(\gamma_0\sigma)}(\overline{w}^1_{\gamma_0\sigma},\overline{w}^2_{\gamma_0\sigma})=\inf_{w^1\in\overline{w}^1_{\gamma_0\sigma},w^2\in\overline{w}^2_{\gamma_0\sigma}}\rho_W(w^1,w^2)$$

is isometric to $W/\upsilon(\gamma_0\sigma)$ and, hence, is a complete metric space as well. If the operator $\Psi:W\to W$ is a Volterra operator on the family υ, then the operator $\Psi_{\gamma_0}:W/\upsilon(\gamma_0)\to W/\upsilon(\gamma_0)$ is a Volterra operator on the family $\overline{\upsilon}_{\gamma_0}$

6. If the sequence $\{\Psi_i:W\to W\}$ of Volterra operators on the family υ converges to the operator $\Psi:W\to W$ (for any $w\in W$ it holds true that $\rho(\Psi_i w,\Psi w)\to 0$), then the operator $\Psi:W\to W$ is a Volterra operator on the family υ as well.

Let us now consider the equation

$$w=\Psi w, \tag{2.2}$$

where $\Psi:W\to W$ is a Volterra operator on the family υ.

D e f i n i t i o n 2. [21] We define a $\upsilon(\gamma)$-*local solution* to the equation (2.2), $\gamma\in(0,1)$, to be a class $\overline{w}_\gamma\in W/\upsilon(\gamma)$ that satisfies the equality $\Psi_\gamma\overline{w}_\gamma=\overline{w}_\gamma$. Identifying the element w that satisfies (2.2) with its class of $\upsilon(1)$-equivalence $\overline{w}$, we consider it a *global solution* to the equation (2.2). We define a $\upsilon(\gamma)$-*maximally extended solution* to the equation (2.2), $\gamma\in(0,1)$, to be a mapping that puts in correspondence to any $\xi\in(0,\gamma)$ the $\upsilon(\xi)$-local solution $\overline{w}_\xi$, and that satisfies the following two conditions:

- for any η,ξ, $0<\eta<\xi<\gamma$, it holds true that $\overline{w}_\xi\subseteq\overline{w}_\eta$ ($\overline{w}_\xi$ is a *restriction* of $\overline{w}_\gamma$);
- for an arbitrary fixed $w^0\in W$, $\lim\limits_{\xi\to\gamma-0} d(w_\xi,w^0_\xi)=\infty$.

D e f i n i t i o n 3. [21] A Volterra operator $\Psi:W\to W$ on the family υ is called *locally contracting at* $\gamma\in[0,1)$, if there exist $q<1$ and $w^0\in\overline{w}_\gamma$ such that for any $r>0$, one can find $\delta>0$ such that for any two $\overline{w}^1_{\gamma+\delta},\overline{w}^2_{\gamma+\delta}\in B_{W/\upsilon(\gamma+\delta)}(\overline{w}^0_{\gamma+\delta},r)$ ($\overline{w}^0_{\gamma+\delta}=\Pi_{\gamma+\delta}w^0$), which, in the case $\gamma>0$, satisfy for any $\xi\in(0,\gamma)$ the inclusion $\overline{w}^1_{\gamma+\delta},\overline{w}^2_{\gamma+\delta}\subset\overline{w}^0_\xi$, where ($\overline{w}^0_\xi=\Pi_\xi w^0$), the following inequality holds true:

$$d(\Psi_{\gamma+\delta}\overline{w}^1_{\gamma+\delta},\Psi_{\gamma+\delta}\overline{w}^2_{\gamma+\delta})\le qd(\overline{w}^1_{\gamma+\delta},\overline{w}^2_{\gamma+\delta}) \qquad (2.3).$$

D e f i n i t i o n 4. [21] A Volterra (on the family υ) operator $\Psi:W\to W$ is called *locally contracting on the family* υ if it is locally contracting at any $\gamma\in[0,1)$ with the constants q and $\delta(r)$, which are independent of the choice of $\gamma\in[0,1)$.

The following theorem on solvability of the equation (2.2) can be formulated.

T h e o r e m 2.1. [21] *Let the Volterra (on the family* υ *) operator* Ψ *be locally contracting on* υ *.*

Then the equation (2.2) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution.*

We introduce now the following version of the equation (2.2)

$$w = F(w, \lambda), \tag{2.4}$$

parameterized by $\lambda \in \Lambda$, where Λ is some metric space. For each $\lambda \in \Lambda$, the operator $F(\cdot,\lambda): W \to$ $\to W$ is a Volterra on the family υ and $F(\cdot,\lambda_0)=\Psi$ for some $\lambda_0 \in \Lambda$. Our aim is to formulate conditions for existence and uniqueness of solutions to the equation (2.4) (we, naturally, apply Definition 2 to the equation (2.4) at each fixed $\lambda \in \Lambda$); and convergence of these solutions to solution to the equation (2.2) as $\lambda \to \lambda_0$. This means, that the problem (2.4) is well-posed.

D e f i n i t i o n 5. For any $\lambda \in \Lambda_0 \subseteq \Lambda$, let the Volterra operator $F(\cdot,\lambda): W \to W$ be given. This family of operators is called *uniformly locally contracting on the family* υ if for each $\lambda \in \Lambda_0 \subseteq \Lambda$, the operator $F(\cdot,\lambda): W \to W$ is locally contracting on υ with the constants q and $\delta(r)$, which are independent of the choice of $\lambda \in \Lambda_0$.

Now we are ready to formulate and prove the following theorem on well-posedness of the equation (2.4).

T h e o r e m 2.2. *Assume that the following two conditions are satisfied:*

1) *There exists* $\varrho_0 > 0$ *such that the operators* $F(\cdot,\lambda): W \to W$ *,* $\lambda \in B_\Lambda(\lambda_0, \varrho_0)$ *are uniformly locally contracting on the family* υ *;*

2) *For an arbitrary* $w \in W$ *, the mapping* $F: W \times \Lambda \to W$ *is continuous at* (w, λ_0) *.*

Then for each $\lambda \in \Lambda_0$ *, the equation* (2.4) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution.*

If the equation (2.4) *has a global solution* $\overline{w}_0 = w_0$ *at* $\lambda = \lambda_0$ *, then for each* λ *(sufficiently close to* λ_0 *) it also has a global solution* $w = w(\lambda)$ *, and* $\rho(w(\lambda), w_0) \to 0$ *as* $\lambda \to \lambda_0$ *.*

If the equation (2.4) *has a maximally extended solution* $\overline{w}_{0\zeta}$ *at* $\lambda = \lambda_0$ *, then for any* $\gamma \in (0, \zeta)$ *one can find a neighborhood of* λ_0 *such that for any* λ *the equation* (2.4) *has a local solution* $\overline{w}_\gamma = \overline{w}_\gamma(\lambda)$ *in this neighborhood and* $d_{W/\upsilon(\gamma)}(\overline{w}_\gamma(\lambda), \overline{w}_{0\gamma}) \to 0$ *as* $\lambda \to \lambda_0$ *.*

P r o o f.

The unique solvability of the equation (2.4) for any $\lambda \in B_\Lambda(\lambda_0, \varrho_0)$ follows from Theorem 1.

We prove the continuous dependence of solutions on a parameter λ. Consider the case when the equation (2.4) has global solution $w_0 = w(\lambda_0) \in W$ at $\lambda = \lambda_0$. Choose an arbitrary $\varepsilon > 0$. Let us find $\delta > 0$ satisfying Definition 3 at $r_1 = \rho(w_0, w^0) + 1$, $\gamma = 0$ and any $\lambda \in B_\Lambda(\lambda_0, \varrho_0)$. For $k = [\frac{1}{\delta}] + 1$ denote $\Delta_l = l\delta$, $l = 1, 2, \ldots, k$. Since the condition 2) holds true, for any $\varepsilon > 0$ one can find $\sigma_1 > 0$ and $\varrho_1 > 0$ such that for each $\lambda \in B_\Lambda(\lambda_0, \varrho_1)$ we have

$$d_W(F(\varpi, \lambda), F(w_0, \lambda_0)) < \frac{(1-q)\varepsilon}{6}$$

for all $\varpi \in B_W(w_0, \sigma_1)$. Assume that $\sigma_1 < \frac{(1-q)\varepsilon}{6}$. Let us find $\sigma_2 > 0$ and ϱ_2 such that for arbitrary $\lambda \in B_\Lambda(\lambda_0, \varrho_2)$ it holds that

$$d_{W/\upsilon(\Delta_{k-1})}(F_{\Delta_{k-1}}(\overline{\varpi}_{\Delta_{k-1}}, \lambda), F_{\Delta_{k-1}}(\overline{w}_{0\Delta_{k-1}}, \lambda_0)) < \frac{(1-q)\sigma_1}{6}$$

for all $\varpi_{\Delta_{k-1}} \in B_{W/\upsilon(\Delta_{k-1})}(\overline{w}_{0\Delta_{k-1}}, \sigma_2)$. Assume that $\sigma_2 < \frac{(1-q)\sigma_1}{6}$, $\varrho_2 \leq \varrho_1$. There exist $\sigma_3 > 0$ and ϱ_3 such that for any $\lambda \in B_\Lambda(\lambda_0, \varrho_3)$ it holds true that

$$d_{W/\upsilon(\Delta_{k-2})}(F_{\Delta_{k-2}}(\varpi_{\Delta_{k-2}}, \lambda), F_{\Delta_{k-2}}(\overline{w}_{0\Delta_{k-2}}, \lambda_0)) < \frac{(1-q)\sigma_2}{6}$$

for any $\varpi_{\Delta_{k-2}} \in B_{W/\upsilon(\Delta_{k-2})}(\overline{w}_{0\Delta_{k-2}}, \sigma_3)$; $\sigma_3 < \frac{(1-q)\sigma_2}{6}$, $\varrho_3 \leq \varrho_2$ etc. We perform k iterations and at the last step find σ_k and ϱ_k, $0 < \sigma_k < \frac{(1-q)\sigma_{k-1}}{6}$, $\varrho_k \leq \varrho_{k-1}$.

Let $\overline{w}_{0\Delta_1}$ denote a $\upsilon(\Delta_1)$-local solution to the equation (2.4) at $\lambda = \lambda_0$, that is a fixed point of the operator $F_{\Delta_1}(\cdot, \lambda_0) : W/\upsilon(\Delta_1) \to W/\upsilon(\Delta_1)$. If $d_{W/\upsilon(\Delta_1)}(\varpi_{\Delta_1}, \overline{w}_{0\Delta_1}) < \sigma_k$, then

$$d_{W/\upsilon(\Delta_1)}(F_{\Delta_1}(\varpi_{\Delta_1}, \lambda), F_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda_0)) < \frac{(1-q)\sigma_{k-1}}{6}$$

for all $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$.

Taking into account the condition 1), we get for any natural number m that

$$d_{W/\upsilon(\Delta_1)}(F^m_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda), w_{0\Delta_1}) \leq d_{W/\upsilon(\Delta_1)}(\overline{w}_{0\Delta_1}, \lambda), F^{m-1}_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda)) + \ldots$$

$$\ldots + d_{W/\upsilon(\Delta_1)}(F_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda), \overline{w}_{0\Delta_1}) \leq (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-1}}{6} \leq \frac{\sigma_{k-1}}{6}.$$

Due to the convergence of the sequential approximations $F^m_{\Delta_1}(\overline{w}_{0\Delta_1}, \lambda)$ to the fixed point $\overline{w}_{\Delta_1} = \overline{w}_{\Delta_1}(\lambda)$ of the operator $F_{\Delta_1}(\cdot, \lambda) : W/\upsilon(\Delta_1) \to W/\upsilon(\Delta_1)$, we obtain that $d_{W/\upsilon(\Delta_1)}(\overline{w}_{\Delta_1}, \overline{w}_{0\Delta_1}) \leq \leq \frac{\sigma_{k-1}}{6}$ for each $\lambda \in U_k$. Further, let $\overline{w}_{0\Delta_2}$ be a $\upsilon(\Delta_2)$-local solution to the equation (2.4) at $\lambda = \lambda_0$. Then, for all $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$, $\varrho_k \leq \varrho_{k-1}$ and any $\varpi_{\Delta_2} \in B_{W/\upsilon(\Delta_2)}(\overline{w}_{0\Delta_2}, \sigma_{k-1}) \bigcap \overline{w}_{\Delta_1}$ we get

$$d_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \overline{w}_{0\Delta_2}) = d_{W/\upsilon(\Delta_1)}(F_{\Delta_2}(u_{\Delta_2}, \lambda), F_{\Delta_2}(\overline{w}_{0\Delta_2}, \lambda_0)) < \frac{(1-q)\sigma_{k-2}}{6}.$$

Then

$$d_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) < \sigma_{k-1} + \frac{(1-q)\sigma_{k-2}}{6} < \frac{(1-q)\sigma_{k-2}}{3}.$$

For all $m = 1, 2, \ldots$ we have

$$d_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) \leq d_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), F^{m-1}_{\Delta_2}(\varpi_{\Delta_2}, \lambda)) + \ldots$$

$$\ldots + d_{W/\upsilon(\Delta_2)}(F_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) \leq (q^{m-1} + \ldots + q + 1)\frac{(1-q)\sigma_{k-2}}{3} \leq \frac{\sigma_{k-2}}{3}.$$

Taking into account the convergence of the approximations $F^m_{\Delta_2}(u_{\Delta_2}, \lambda)$ to $\overline{w}_{\Delta_2} = \overline{w}_{\Delta_2}(\lambda)$ we obtain

$$d_{W/\upsilon(\Delta_2)}(\overline{w}_{\Delta_2}, \overline{w}_{0\Delta_2}) \leq d_{W/\upsilon(\Delta_2)}(\overline{w}_{\Delta_2}, F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda)) +$$

$$+ d_{W/\upsilon(\Delta_2)}(F^m_{\Delta_2}(\varpi_{\Delta_2}, \lambda), \varpi_{\Delta_2}) + d_{W/\upsilon(\Delta_2)}(\varpi_{\Delta_2}, \overline{w}_{0\Delta_2}) \leq \frac{\sigma_{k-2}}{3} + \sigma_{k-1} \leq \frac{\sigma_{k-2}}{2}.$$

Using the convergence of the sequential approximations $F^m_{\Delta_3}(\varpi_{\Delta_3}, \lambda)$ to a fixed point $\overline{w}_{\Delta_3} = \overline{w}_{\Delta_3}(\lambda)$ of the operator $F_{\Delta_3}(\cdot, \lambda) : W/\upsilon(\Delta_3) \to W/\upsilon(\Delta_3)$ for any $\varpi_{\Delta_3} \in B_{W/\upsilon(\Delta_3)}(\overline{w}_{0\Delta_3}, \sigma_{k-2}) \bigcap \overline{w}_{\Delta_2}$ and each $\lambda \in B_\Lambda(\lambda_0, \varrho_k)$, $\varrho_k \leq \varrho_{k-1}$, we obtain the estimate $d_{W/\upsilon(\Delta_3)}(\overline{w}_{\Delta_3}, \overline{w}_{0\Delta_3}) \leq \frac{\sigma_{k-3}}{2}$. We, then, repeat this procedure. At the k-th step we prove in an analogous way that the inequality $\rho_W(w(\lambda), w_0) < \varepsilon$ holds true for all $\lambda \in U_k$. Therefore, $\rho_W(w(\lambda), w_0) \to 0$ as $\lambda \to \lambda_0$.

Let now a solution $\overline{w}_{0\eta}$ to the equation (2.4) at $\lambda = \lambda_0$ be maximally extended. Fix arbitrary $\gamma \in (0, \eta)$ and let $\overline{w}_{0\gamma}$ denote the restriction of the solution $\overline{w}_{0\eta}$. For the equation $\varpi_\gamma = F_\gamma(\varpi_\gamma, \lambda_0)$ the element $\overline{w}_{0\gamma} \in W/\upsilon(\gamma)$ is a global solution. As is shown above, for all λ from some neighborhood of λ_0 the equations $\overline{\varpi}_\gamma = F_\gamma(\overline{\varpi}_\gamma, \lambda)$ have global solutions $\overline{w}_\gamma(\lambda)$, and $d_{W/\upsilon(\gamma)}(\overline{w}_\gamma(\lambda), \overline{w}_{0\gamma}) \to 0$ as $\lambda \to \lambda_0$. □

3. Existence, uniqueness and continuous dependence on impulsive control of solutions to generalized neural field equations

We assume that the following conditions on the system (1.3), (1.2) are imposed:

(i) For any $t \in [a, \infty)$, $w \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, the function $f(t, \cdot, x, \cdot, w) : [a, t] \times \mathbb{R}^m \to \mathbb{R}^n$ is measurable.

(ii) For almost all $(s, y) \in [a, \infty) \times \mathbb{R}^m$, the function $f(\cdot, s, \cdot, y, \cdot) : [s, \infty) \times \mathbb{R}^m \times \mathbb{R}^n$ is continuous.

(iii) For any $b \in (a, \infty)$ and any $r > 0$, $w \in B_{\mathbb{R}^n}(0, r)$, it holds true that

$$\lim_{\mathfrak{r} \to \infty} \sup_{t \in [a,b],\, x \in B^{out}_{\mathbb{R}^m}(0, \mathfrak{r})} \Big| \int_a^t \int_{\mathbb{R}^m} f(t, s, x, y, w) dy ds \Big| = 0.$$

(iv) There is a non-decreasing function $K : (a, \infty) \to \mathbb{N}_0$ such that for any control $\mathfrak{U}$ and any $b \in (a, \infty)$ it holds true that $\mathrm{card}\big(\{t_k \bigcap [a, b]\}\big) \le K(b)$, where the symbol $card(\cdot)$ denotes the cardinality of the corresponding set.

(v) $u_k \in C_0(\mathbb{R}^m, \mathbb{R}^n)$ for all $k \in \mathrm{N}$.

By the virtue of the assumptions made, the element w, defined by (1.3), (1.2), can be represented as

$$w(t, x) = v(t, x) + \sum_{k=1}^{\infty} \chi_{[t_k, \infty)}(t) u_k(x), \tag{3.1}$$

where the function $v : [a, \infty) \to C_0(\mathbb{R}^m, \mathbb{R}^n)$ belongs to locally convex space with topology of uniform convergence on the each compact $[a, b]$ (see [20]). We naturally denote this space as $C([a, \infty), C_0(\mathbb{R}^m, \mathbb{R}^n))$. The space $C([a, b], C_0(\mathbb{R}^m, \mathbb{R}^n))$ of restrictions on $[a, b]$ of the functions from $C([a, \infty), C_0(\mathbb{R}^m, \mathbb{R}^n))$ is a Banach space with the norm $\|v\|_{C([a,b], C_0(\mathbb{R}^m, \mathbb{R}^n))} = \max_{t \in [a,b]} \|v(t)\|_{C_0(\mathbb{R}^m, \mathbb{R}^n)}$.

We define the set $W([a, \infty)) = W([a, \infty), C_0(\mathbb{R}^m, \mathbb{R}^n))$ of the mappings $w : [a, \infty) \to C_0(\mathbb{R}^m, \mathbb{R}^n)$ of the kind (3.1), where v is an arbitrary element of $C([a, \infty), C_0(\mathbb{R}^m, \mathbb{R}^n))$, $\{(t_k, u_k), k \in \mathbb{N}\}$ is any set satisfying the conditions (iv) and (v). We denote $W([a, b]) = W([a, b], C_0(\mathbb{R}^m, \mathbb{R}^n))$ to be the set of restrictions on $[a, b]$ of the functions from $W([a, \infty))$. Choose an arbitrary $b \in (a, \infty)$.

L e m m a 3.1. The set $W([a, b]) = W([a, b], C_0(\mathbb{R}^m, \mathbb{R}^n))$ is complete metric space with respect to the metric

$$\rho_{W([a,b])}(w^1, w^2) =$$

$$= \|v^1 - v^2\|_{C([a,b], C_0(\mathbb{R}^m, \mathbb{R}^n))} + \int_a^b \Big\| \sum_{k:\, t_k^1 \in [a,b]} \chi_{[t_k^1, b]}(t) u_k^1 - \sum_{k:\, t_k^2 \in [a,b]} \chi_{[t_k^2, b]}(t) u_k^2 \Big\|_{C_0(\mathbb{R}^m, \mathbb{R}^n)} dt.$$

P r o o f. For the sake of convenience, we consider the element $U(\cdot, x)$ in the expression $w(t, x) = v(t, x) + U(t, x)$, $U(t, x) = \sum_{k=0}^{K(b)} \chi_{[t_k, b]}(t) u_k(x)$, $x \in \mathbb{R}^m$, as the mapping $t \in [a, b] \mapsto U(t, \cdot) \in C_0(\mathbb{R}^m, \mathbb{R}^n)$. This mapping is piece-wise continuous, continuous from the right and has not more than $K(b)$ discontinuity points of the first kind. The set of such functions U is a subset of the Banach space $L([a, b], \mu, C_0(\mathbb{R}^m, \mathbb{R}^n))$ of measurable summable functions $\Upsilon : [a, b] \to C_0(\mathbb{R}^m, \mathbb{R}^n)$ with the norm $\|\Upsilon\|_{L([a,b], \mu, C_0(\mathbb{R}^m, \mathbb{R}^n))} = \int_a^b \|\Upsilon(t)\|_{C_0(\mathbb{R}^m, \mathbb{R}^n)} dt$. In order to prove the completeness of the space $W([a,b])$ it suffices to show that the set of piece-wise continuous, continuous from the right functions $U:[a,b]\to C_0(\mathbb{R}^m,\mathbb{R}^n)$ having not more than $K(b)$ discontinuity points of the first kind is complete with respect to the metric of $L([a,b],\mu,C_0(\mathbb{R}^m,\mathbb{R}^n))$.

We will say that ξ is an essential value of the mapping $U:[a,b]\to C_0(\mathbb{R}^m,\mathbb{R}^n)$ if there exists $t_0\in[a,b]$ such that $\xi=U(t_0)$ and for any $\varepsilon>0$ one can find $\delta>0$ such that $\|U(t_0)-U(t)\|_{C_0(\mathbb{R}^m,\mathbb{R}^n)}<\varepsilon$ for all $t\in B_{\mathbb{R}}(t_0,\delta)$.

We note that the mapping $\mathcal{U}:[a,b]\to C_0(\mathbb{R}^m,\mathbb{R}^n)$ having N essential values can not be the limit in the metric of $L([a,b],\mu,C_0(\mathbb{R}^m,\mathbb{R}^n))$ of the sequence of functions having not more than $N-1$ essential values. Indeed, as the values $u_1,\dots,u_N$ are essential, there exists a finite set of the semi-intervals $T_i^{\mathrm{l}_i}$, providing partition of $[a,b]$ such that $\mathcal{U}(t)=u_i$ for almost all $t\in T_i^{\mathrm{l}_i}$, $1\le \mathrm{l}_i\le \mathrm{L}_i<\infty$. Choosing $\varepsilon<\widetilde{T}d$, where $\widetilde{T}=\min\limits_{1\le \mathrm{l}_i\le \mathrm{L}_i,\ 1\le i\le N}\mu(T_i^{\mathrm{l}_i})$, $d=\min\limits_{1\le i\le N-1}\|u_i-u_{i+1}\|_{C_0(\mathbb{R}^m,\mathbb{R}^n)}$, we get $\int\limits_a^b\|\mathcal{U}(t)-U(t)\|_{C_0(\mathbb{R}^m,\mathbb{R}^n)}dt>\varepsilon$ for any mapping $U:[a,b]\to C_0(\mathbb{R}^m,\mathbb{R}^n)$, having not more than $N-1$ essential values. From the given proof it also follows that the piece-wise constant continuous from the right mappings $\mathcal{U}:[a,b]\to C_0(\mathbb{R}^m,\mathbb{R}^n)$, $\mathcal{U}(t)=\sum\limits_{k=0}^{\mathrm{K}}\chi_{[t_k,b]}(t)u_k$, $u_k\in C_0(\mathbb{R}^m,\mathbb{R}^n)$, $k=1,\dots,\mathrm{K}$, having K discontinuity points ($\mathrm{K}=\sum\limits_{k=1}^{N}\mathrm{l}_i$, $u_k\not\equiv 0$, $k=1,\dots,\mathrm{K}$) can not be the limit (in the metric of $L([a,b],\mu,C_0(\mathbb{R}^m,\mathbb{R}^n))$) of the sequence of piece-wise constant continuous from the right mappings having not more than $K-1$ discontinuity points of the first kind.

Thus, the limit of the sequence of piece-wise constant continuous from the right mappings having not more than $K(b)$ discontinuity points of the first kind in the chosen metric is a class of mappings (element of $L([a,b],\mu,C_0(\mathbb{R}^m,\mathbb{R}^n))$) including a piece-wise constant continuous from the right having not more than $K(b)$ discontinuity points of the first kind. Hence, the completeness of $W([a,b],C_0(\mathbb{R}^m,\mathbb{R}^n))$ is proved.

D e f i n i t i o n 3.1. We define a local solution to the system (1.3), (1.2), defined on $[a,a+\gamma]$, $\gamma\in(0,\infty)$, to be a mapping $w_\gamma\in W([a,a+\gamma])$, that satisfies the equation

$$w(t,\cdot)=\int\limits_a^t\int\limits_{\mathbb{R}^m}f(t,s,\cdot,y,w(s,y))dyds+\sum_{k=0}^{\infty}\chi_{[t_k,\infty)}(t)u_k(\cdot)$$

on $[a,a+\gamma]$. We define a maximally extended solution to the system (1.3), (1.2), defined on $[a,a+\eta)$, $\eta\in(0,\infty)$, to be a mapping $w_\eta:[a,a+\eta)\to C_0(\mathbb{R}^m,\mathbb{R}^n)$, whose restriction w_γ on $[a,a+\gamma]$ for any $\gamma<\eta$ is its local solution and $\lim\limits_{\gamma\to\eta-0}\rho_{W([a,a+\gamma])}(w_\gamma,0)=\infty$ (here $\mathsf{o}\in W([a,a+\gamma])$ is the identical zero function). We define a global solution to the system (1.3), (1.2) to be a mapping $w:[a,\infty)\to C_0(\mathbb{R}^m,\mathbb{R}^n)$, whose restriction w_γ on $[a,a+\gamma]$ for any $\gamma\in(0,\infty)$ is its local solution.

T h e o r e m 3.1. *Let the assumptions* (i) – (v) *hold true. Assume that for any* $b\in(a,\infty)$, $r>0$ *there exists such integrable on* $[a,b]\times\mathbb{R}^m$ *function* $\widetilde{f}_r$ *that* $|f(t,s,x,y,w^1)-f(t,s,x,y,w^2)|\le\widetilde{f}_r(s,y)|w^1-w^2|$ *for all* $w^1,w^2\in B_{\mathbb{R}^n}(0,r)$, *almost all* $(s,y)\in[a,b]\times\mathbb{R}^m$ $t\in[a,\infty)$, $x\in B_{\mathbb{R}^m}(0,r)$ *and. Then, for each set of points* $(t_k,u_k)\in[a,\infty)\times C_0(\mathbb{R}^m,\mathbb{R}^n)$, $k\in\mathbb{N}$, *the system* (1.3), (1.2) *has a unique global or maximally extended solution, and each local solution is a restriction of this solution.*

Moreover, assume that for some set of points $(t_k,u_k)\in[a,\infty)\times C_0(\mathbb{R}^m,\mathbb{R}^n)$, $k\in\mathbb{N}$ *and some sequences* $\{(t_k^i,u_k^i)\}_{i=1}^{\infty}\subset[a,\infty)\times C_0(\mathbb{R}^m,\mathbb{R}^n)$, $k\in\mathbb{N}$, *for any* $b>a$, *it holds true that*

$$\int\limits_a^b\max_{x\in\mathbb{R}^m}\Big|\sum_{k:\,t_k^i\in[a,b]}\chi_{[t_k^i,\infty)}u_k^i(x)-\sum_{k:\,t_k^0\in[a,b]}\chi_{[t_k^0,\infty)}u_k^0(x)\Big|dt\to 0\text{ as }i\to\infty.$$

Then, if for the control $\{(t_k^0, u_k^0) \in [a,\infty) \times C_0(\mathbb{R}^m, \mathbb{R}^n)\,,\ k \in \mathbb{N}\}$ *, the problem* (1.3) *,* (1.2) *has a local solution* w_γ^0 *, defined on* $[a, a+\gamma]$ *, one can find a number* I *, such that for all* $i > I$ *the problem* (1.3) *,* (1.2) *with the control* $\{(t_k^i, u_k^i) \in [a,\infty) \times C_0(\mathbb{R}^m, \mathbb{R}^n)\,,\ k \in \mathbb{N}\}$ *, has a local solution* w_γ^i *, defined on* $[a, a+\gamma]$ *, and* $\rho_{W([a,a+\gamma])}(w_\gamma^i, w_\gamma^0) \to 0$ *as* $i \to \infty$ *.*

P r o o f. We are going to apply Theorem 2.2, so we parameterize (1.3) with respect to the control imposed and represent the result in terms of the following operator equation

$$w = F(w, \lambda), \tag{3.2}$$

$$\big(F(w,\lambda)\big)(t,x) = \int_a^t \int_{\mathbb{R}^m} f(t,s,x,y,w(s,y))dyds + \sum_{k=0}^{\infty} \chi_{[t_{\lambda k},\infty)}(t) u_{\lambda k}(x).$$

Choose arbitrary $b \in (a,\infty)$. For the chosen b, we define the relation $\upsilon(\sigma)$, $\sigma \in [0,1]$, on $W([a,b]) = W([a,b], C_0(\mathbb{R}^m, \mathbb{R}^n))$ as follows:

$$(w^1, w^2) \in \upsilon(\sigma) \Longleftrightarrow w^1(t,\cdot) = w^2(t,\cdot)\ t \in [0, (b-a)\sigma].$$

Choose arbitrary $q<1$, $r>0$. Let $\gamma \in (0, b-a)$ and $w^1(t,\cdot) = w^2(t,\cdot)$, $t \in [a, a+\gamma]$, where $w^1, w^2 \in B_{C([a,b],C_0(\mathbb{R}^m,\mathbb{R}^n))}(0,r)$ (in the notation of the ball, 0 denotes the function that is identically equal to zero). Using assumptions (i)–(v) and condition 1) of Theorem 3.1, we get the following estimates

$$\rho_{W([a,a+\gamma+\delta])}(F(w^1,\lambda), F(w^2,\lambda)) =$$

$$\max_{t\in[a,a+\gamma+\delta],x\in\mathbb{R}^m} \Big| \int_a^t \int_{\mathbb{R}^m} f(t,s,x,y,w^1(s,y))dyds -$$

$$- \int_a^t \int_{\mathbb{R}^m} f(t,s,x,y,w^2(s,y))dyds \Big| \leq$$

$$\varepsilon/2 + \max_{t\in[a,a+\gamma+\delta],x\in B_{\mathbb{R}^m}(0,r_\varepsilon)} \int_{a+\gamma}^{a+\gamma+\delta} \int_{\mathbb{R}^m} \Big| f(t,s,x,y,w^1(s,y))dyds -$$

$$- f(t,s,x,y,w^2(s,y)) \Big| dyds \leq$$

$$\varepsilon/2 + \max_{t\in[a,a+\gamma+\delta],x\in B_{\mathbb{R}^m}(0,r_\varepsilon)} \Big| \int_{a+\gamma}^{a+\gamma+\delta} \int_{\mathbb{R}^m} \widetilde{f}_r(s,y) \|w^1 - w^2\|_{C([a,b],BC(\mathbb{R}^m,\mathbb{R}^n))} dyds \Big| \leq \varepsilon.$$

Here, $r_\varepsilon > 0$, $\delta > 0$ can be chosen in a such way that $\varepsilon < q$. Thus, we checked that the inequality (2.3) is satisfied. By the virtue of arbitrary choice of $b \in (a,\infty)$, using Theorem 2.1, we prove existence of a unique global (if the distances of the solutions obtained to the element $0 \in W$ are uniformly bounded for all $b \in (a,\infty)$) or maximally extended (otherwise) solution to (3.2) and, hence, to (1.2), (1.3).

Next, we take arbitrary $\varepsilon > 0$, $\widehat{w} \in W([a,b], C_0(\mathbb{R}^m,\mathbb{R}^n))$, $w^i \subset W([a,b], C_0(\mathbb{R}^m,\mathbb{R}^n))$, $\rho_{W([a,b])}(w^i, \widehat{w})$ ($i \to \infty$). For the chosen sequences $\{t_k^i\} \subset [a,\infty)$, $\{u_k^i\} \subset C_0(\mathbb{R}^m,\mathbb{R}^n)$, such that $t_k^i \to t_k^0$ and $\|u_k^i - u_k^0\|_{C_0(\mathbb{R}^m,\mathbb{R}^n)} \to 0$ for each k as $i \to \infty$, we estimate

$$\rho_{W([a,b])}(F(\widehat{w}, 0), F(w^i, 1/i)) \leq$$

$$\leq \max_{t\in[a,b],x\in\mathbb{R}^m} \int\limits_a^t \int\limits_{\mathbb{R}^m} \Big| f(t,s,x,y,\widehat{w}(s,y)) - f(t,s,x,y,w^i(s,y)) \Big| dyds +$$

$$+ \int\limits_a^b \max_{x\in\mathbb{R}^m} \Big| \sum_{k=0}^{K^0(b)} \chi_{[t_k^0,\infty)}(t) u_k^0(x) - \sum_{k=0}^{K^i(b)} \chi_{[t_k^i,\infty)}(t) u_k^i(x) \Big| dt.$$

Estimation of the integrand $\big|f(t,s,x,y,\widehat{w}(s,y)) - f(t,s,x,y,w^i(s,y))\big|$ of the first summand on the right-hand side using the condition 1) of Theorem 3.1, gives uniform convergence of this expression to 0 on $[a,b]\times\mathbb{R}^m$.

The condition 2) of Theorem 3.1 guarantees convergence of the second summand on the right-hand side of the inequality to 0 as $i\to\infty$.

Thus the verification of the second condition of Theorem 2.2 is complete, and Theorem 3.1 is proved.

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О КОРРЕКТНОСТИ ОБОБЩЕННЫХ УРАВНЕНИЙ НЕЙРОПОЛЕЙ С ИМПУЛЬСНЫМ УПРАВЛЕНИЕМ

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Формулируется и доказывается теорема о корректности абстрактных уравнений Volterra в метрических пространствах. Далее рассматривается нелинейное интегральное уравнение Volterra, частными случаями которого являются уравнения, используемые в математической нейробиологии. Исследуются решения, стремящиеся к нулю в любой момент времени при неограниченном росте пространственной переменной. В литературе такие решения называют «локализованными в пространстве» или «бампами», они соответствуют нормальному функционированию головного мозга. Ставится задача импульсного управления, управляющими параметрами являются моменты времени, в которые решение терпит разрывы, и величины соответствующих скачков решения. Такие управления моделируют электрическую стимуляцию мозга, применяемую при лечении расстройств центральной нервной системы. Для исследования данной управляемой интегральной системы определяется специальное полное метрическое (не являющееся линейным) функциональное пространство. В этом пространстве получены условия существования, единственности и продолжаемости решения, а также его непрерывной зависимости от импульсного управления.
Ключевые слова: абстрактные уравнения Volterra; нелинейные интегральные уравнения; уравнения Volterra; уравнения нейрополей; импульсное управление; корректность.

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